Chow quotients of complete flags

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Quotients by torus actions

GIT quotients

- SETUP: H torus, X projective, normal (usually smooth), $H \curvearrowright X$
- Choose L ample on X, linearize the action on L
- X^H fixed points. To every fixed point component Y we get $\mu_L(Y) \in M(T)$
- Weight polytope P of (X, L): convex hull of the $\mu_L(Y)$
- Up to changing L by a multiple, P is also the polytope of weights on $H^0(X, L)$.
- GIT quotients of (X, L):

$$\mathcal{G}X_v = \operatorname{Proj}\left(\bigoplus_{m\geq 0} H^0(X, mL)_{mv}\right), \text{ for } v \in P$$

- It works on the quasiprojective setting, but P is not necessarily bounded

The Chow quotient

- GIT quotients depends on polarizations/linearizations
- Kapranov: there exists an open subset s.t. closures in X of H-orbits of points of U have the same dimension and belong to the same homology class; after shrinking U, we get a morphism

$$U/H \to \mathrm{Chow}(X)$$

- Chow quotient of X ($\mathcal{C}X$) := (normalization of) closure of its image

The limit quotient

- The GIT quotients of X form an inverse system: if $X^{ss}_{v_1} \subset X^{ss}_{v_2}$ we get a map $\mathcal{G}X_{v_1} \to \mathcal{G}X_{v_2}$
- In the inverse limit of this system there is a unique component dominating all the GIT quotients: its normalization $\mathcal{L}X$ is call the limit quotient of X
- ([BHK]) $\mathcal{C}X$ equals $\mathcal{L}X$.
- It is enough to consider GIT quotients of (X, L)
- $\mathcal{C}X$ is a resolution of the set of birational transformations among geometric quotients of X

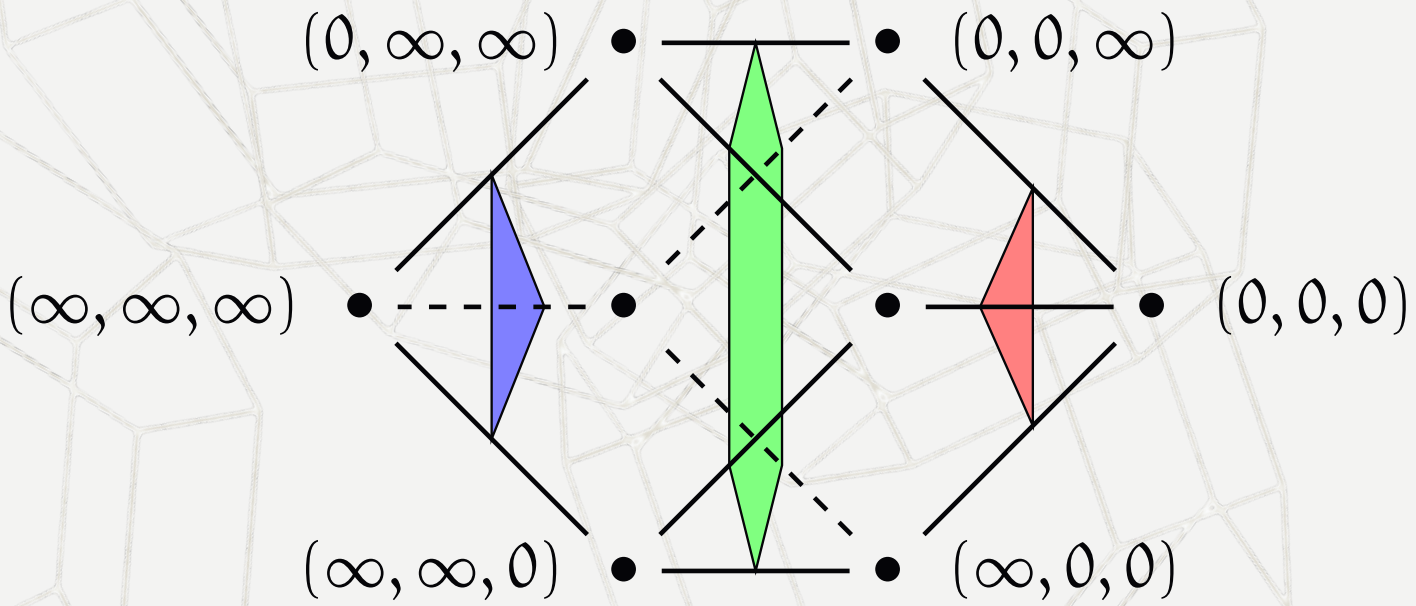
The toric setting (KSZ)

- Case: $X = X(\Sigma) = \overline{T}$ (projective) toric, normal, $H \subset T$ subtorus
- P moment polytope of (X, L)
- GIT quotients are T/H-toric varieties, whose moment polytopes are linear sections of P:

$$\pi : M(T) \to M(H), \text{ take } \pi^{-1}(\nu) \cap P$$

- $M^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow (M^*(T)/\, M^*(H)) \otimes_{\mathbb{Z}} \mathbb{R}$ associated projection
- Consider the quotient fan $\overline{\Sigma}$ (intersections of projections of cones), then $\mathcal{C}X = X(\overline{\Sigma})$
- The moment polytope of $\mathcal{C}X$ is the Minkowski sum of the moment polytopes of the GIT quotients
- If X is not projective, the same procedure provides the limit quotient (CMcL)

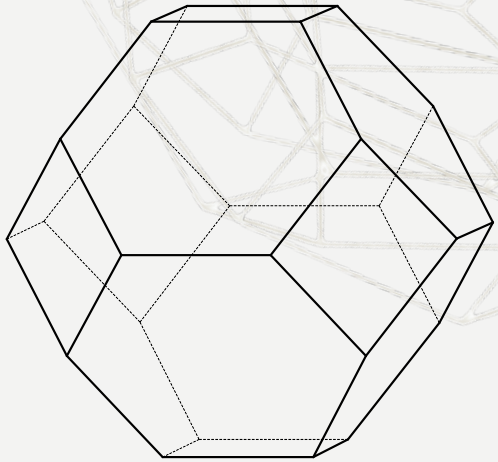
Example 1: $\mathbb{C}^* \curvearrowright X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

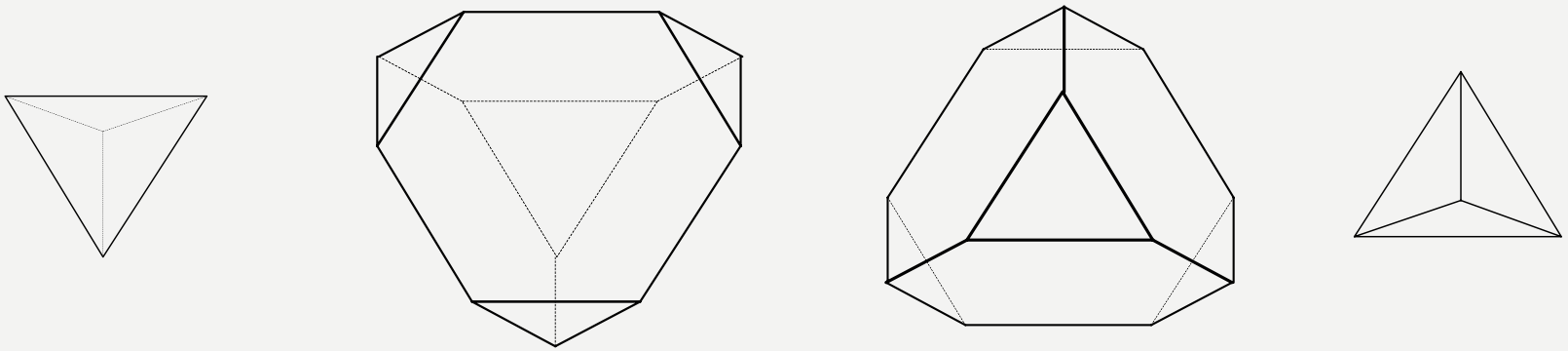


$$\mathcal{G}X(0,1) \longleftarrow \mathcal{G}X(1,2) \longrightarrow \mathcal{G}X(2,3)$$

Example 2: $\mathbb{C}^* \curvearrowright X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

$\mathcal{C}X$ = Losev–Manin 3-fold, whose moment polytope is the permutohedron





RH varieties

$X = G/P$, G semisimple (adjoint), $H \subset B \subset P$ maximal torus.

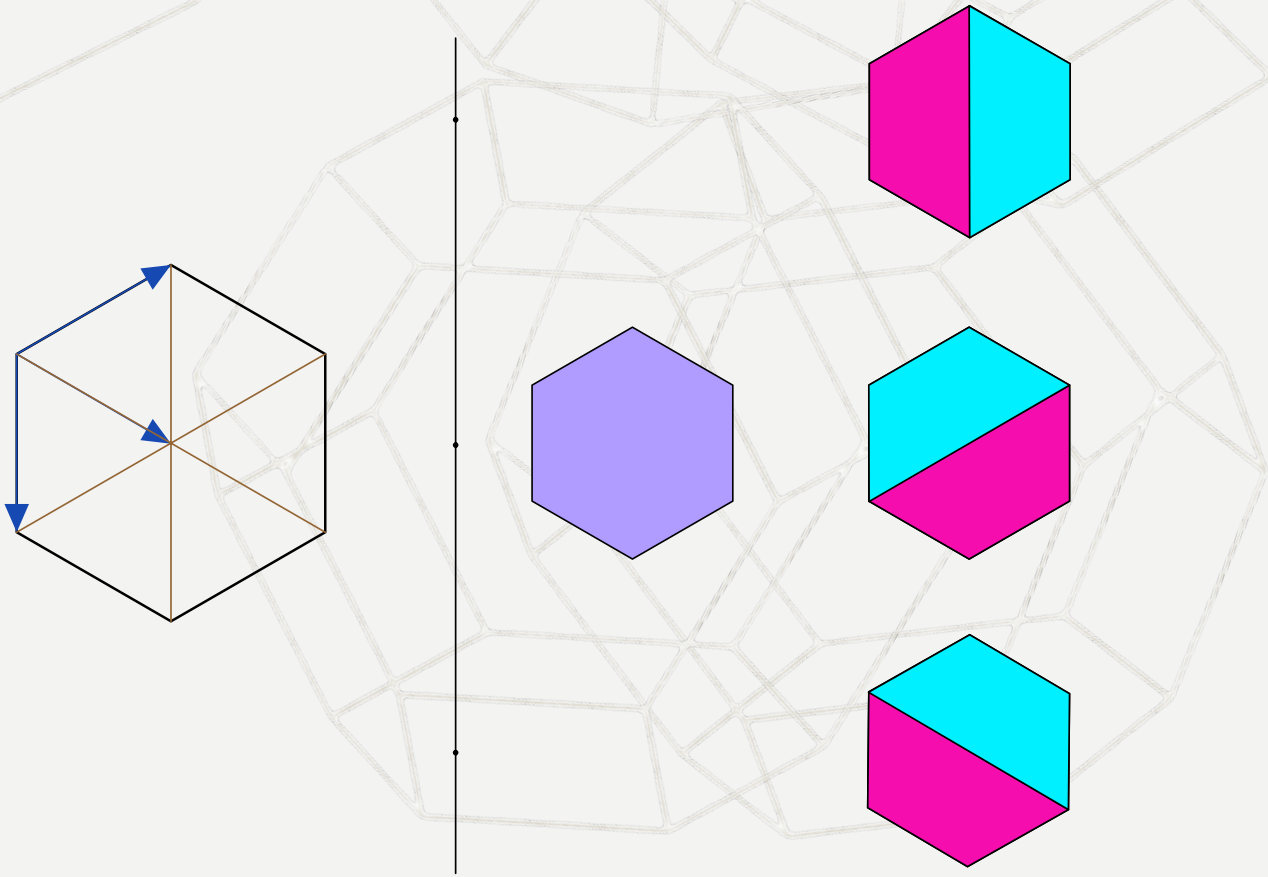
- Kapranov studied the case of Grassmannian of lines in $\mathbb{P}^n$ and maximal torus
- It is the Grothendieck–Knudsen moduli space $\overline{M_{0,n}}$
- Thaddeus: $\mathbb{C}^*$-actions on Grassmannians
- Problem: Study Chow quotients of RH varieties by the maximal torus action
- Observation: $\mathcal{C}(G/P)$ is a contraction of $\mathcal{C}(G/B)$
- It supports the action of $W = N_G(H)/H$

Maximal torus action on $X = G/B$

- $X^H = \{wB |\ w \in W\}$
- If $L = L(\lambda)$, then $\mu_L(wB) = -w(\lambda)$
- H-action on $T_{X,wB} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-w(\alpha)}$
- The general cycle of $\mathcal{C}X$ is a projective variety whose moment polytope is P
- Special cycles correspond to partitions of P

Example: $G = PGl(3)$

$X = \mathbb{P}(T_{\mathbb{P}^2})$, $\mathcal{G}X = \mathbb{P}^1$ (three distinguished points)



Results

We have descriptions in the cases:

- A_3 (complete flag variety of $\mathbb{P}^3$); note [CO]
- C_2 (isotropic flag variety of $\mathbb{P}^3$)
- G_2 (in progress)
- Some RH varieties of Picard number one

In collaboration with L. Barban, M. Bianco, G. Occhetta

Strategy

- It potentially works on smooth projective varieties endowed with an action which is definite on every fixed point
- This essentially means that fixed points are isolated and their weights are all extremal in P
- (this holds for the maximal torus action on every RH variety)
- Under these assumptions:
  1. X is covered by H-invariant open affine spaces $U_x \simeq T_{X,x}$, $x \in X^H$
  2. $\mathcal{C}U_x$ is projective
  3. $\mathcal{C}X$ is the inverse limit of the $\mathcal{C}U_x$

Ideas

- The $\mathcal{C}\mathcal{U}_\chi$ contain local information on the way in which GIT quotients ensemble
- In the RH setting, all the $\mathcal{C}\mathcal{U}$'s are isomorphic

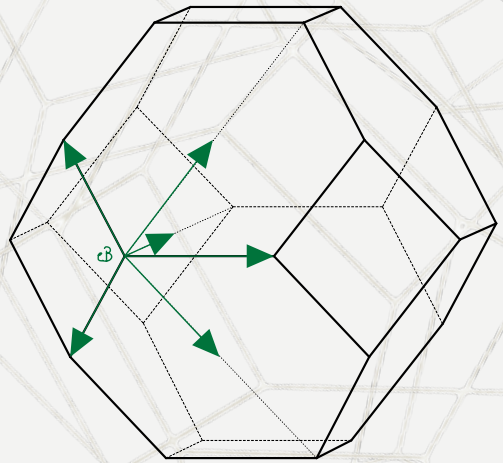
$$\begin{array}{ccccc} \mathcal{U}_e & \xrightarrow{\sigma} & \mathcal{U}_\sigma & \xrightarrow{\text{int}} & \mathcal{U}_e \\ \downarrow & & & & \downarrow \\ \mathcal{C}\mathcal{U}_e & & \dashrightarrow & & \mathcal{C}\mathcal{U}_e \end{array}$$

- So W acts on $\mathcal{C}\mathcal{U}_e$ as a group of birational automorphisms
- This action can be effectively computed in many cases

The Chow quotient of the complete flag of $\mathbb{P}^3$

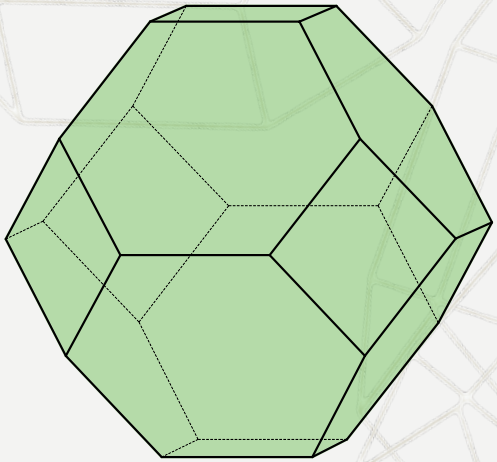
Case A_3

- X is the complete flag variety of $\mathbb{P}^3$
- $\dim(X) = 6$, $\dim(\mathcal{C}X) = 3$, 24 H-fixed points wB, $w \in W$
- P = 3-dimensional permutohedron
- $W = S_4$, but we have another automorphism that survives quotienting by H: anti-transposition τ. Consider $W(\tau) \simeq S_4 \rtimes \mathbb{Z}/2\mathbb{Z}$



Closure of the general orbit

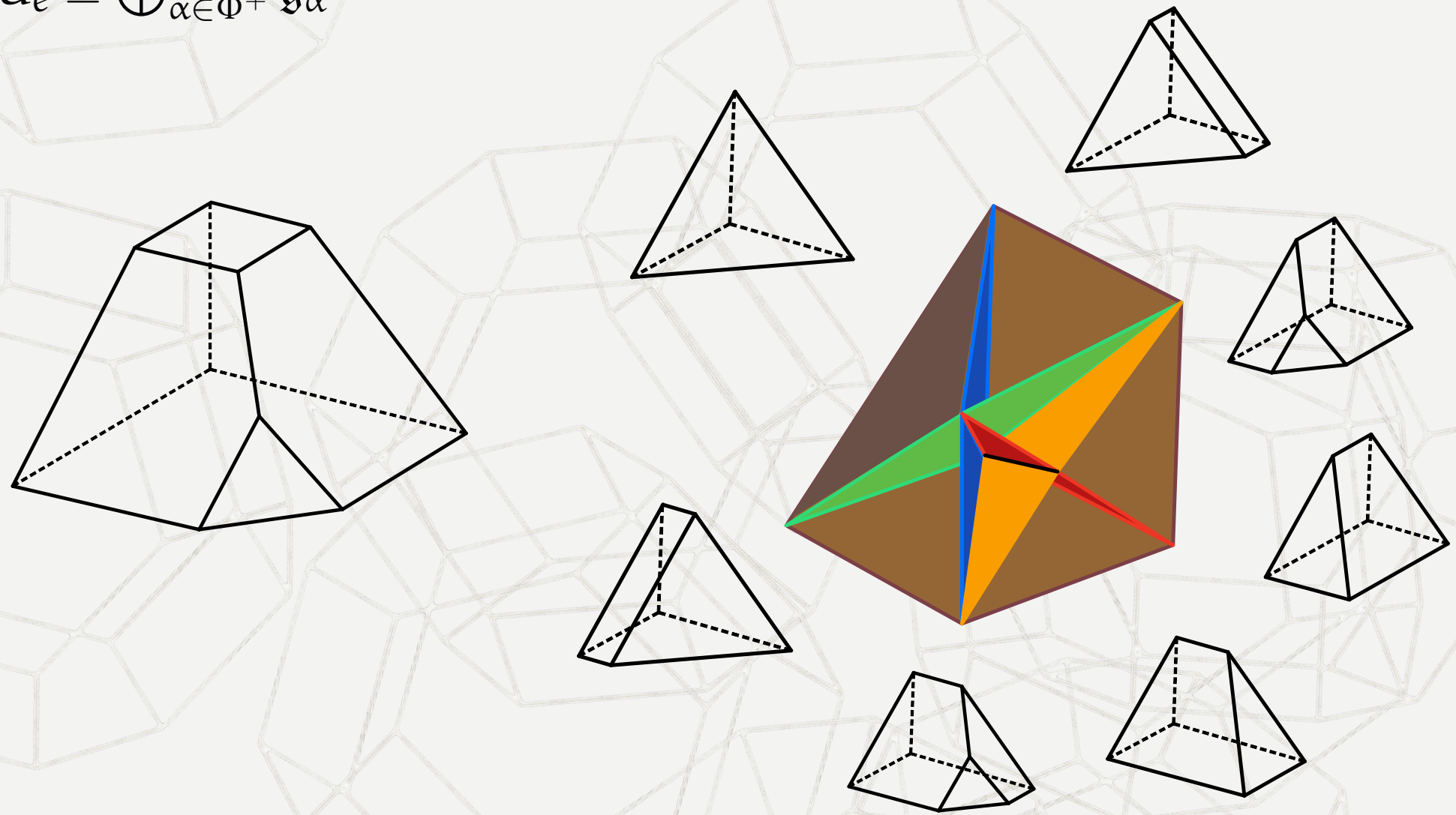
- This is the projective toric variety associated to the 3-dimensional permutohedron P



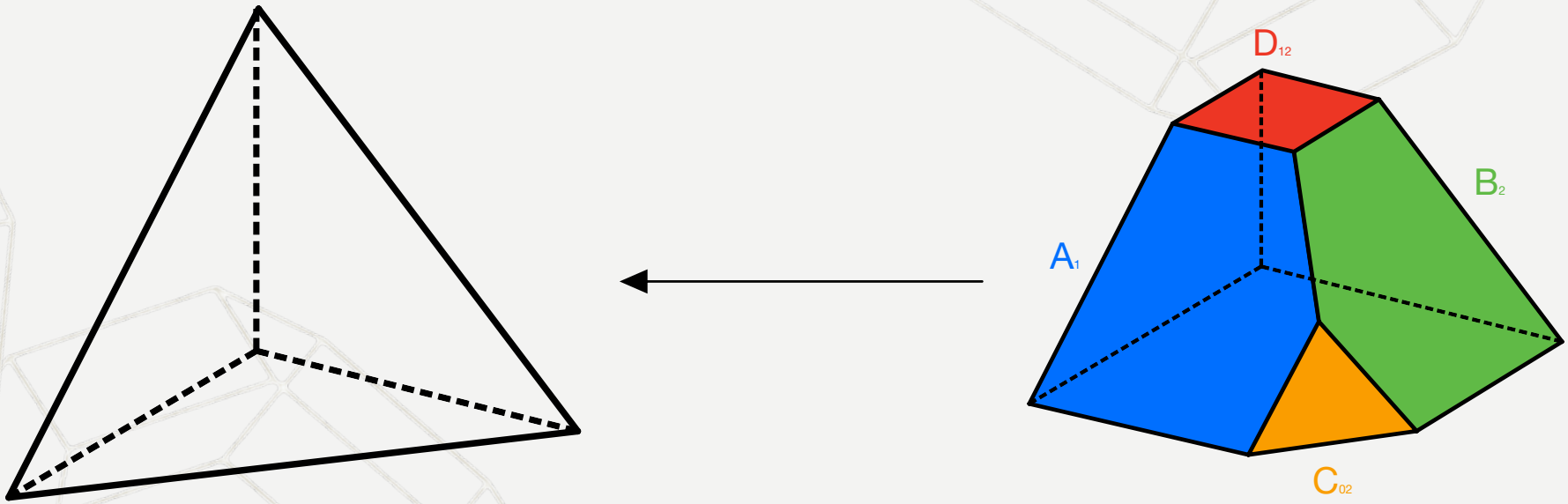
- Its degenerations in $\mathcal{C}\mathsf{X}$ are unions of toric H-varieties whose moment polytopes are given by subdivisions of P

The limit quotient of $\mathcal{U}_e$

$\mathcal{U}_e \simeq \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$



The boundary of $\mathcal{CU}_e$



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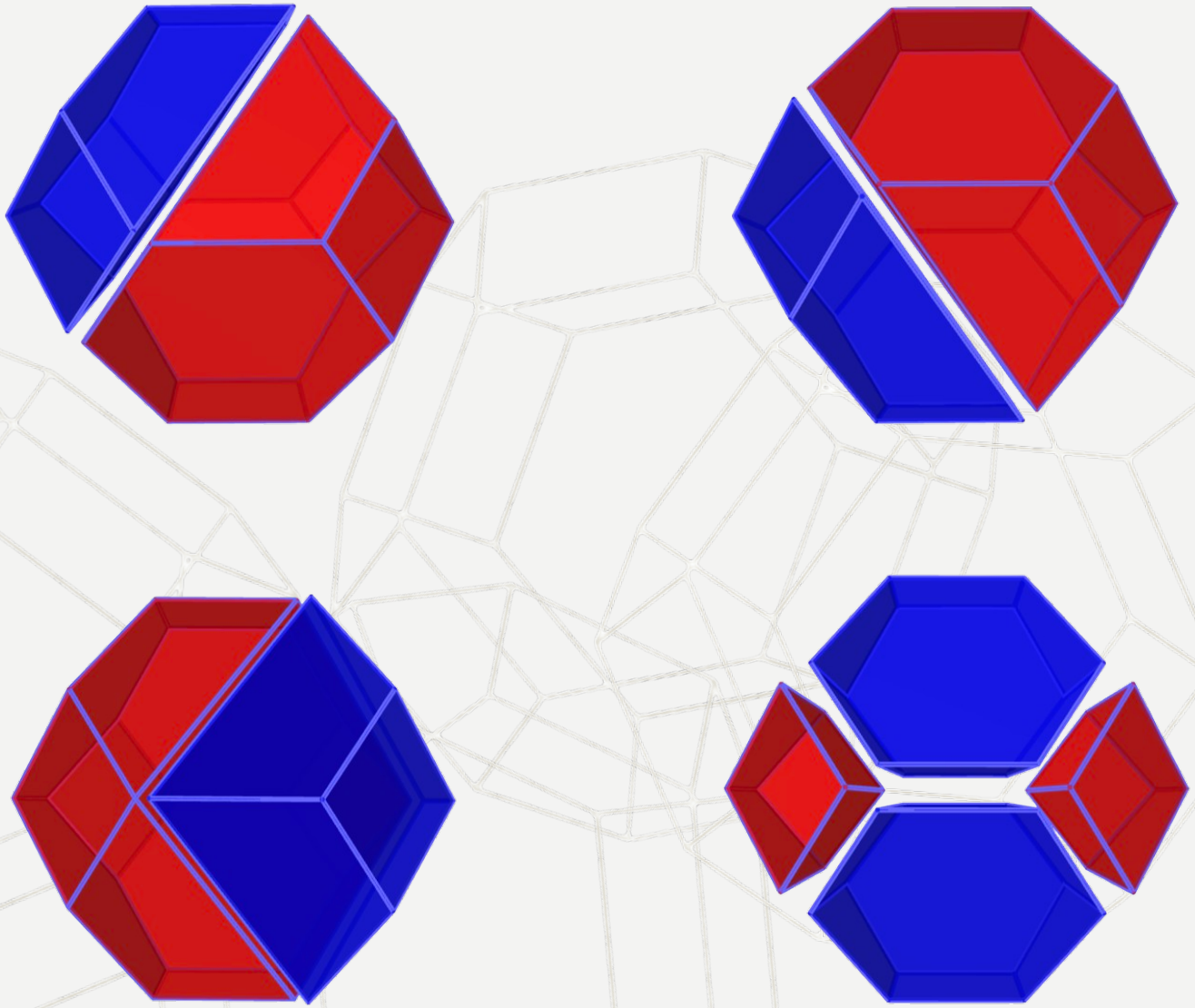
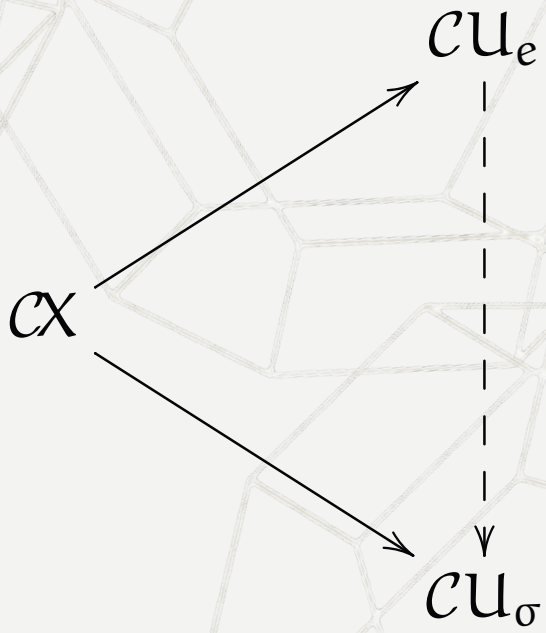
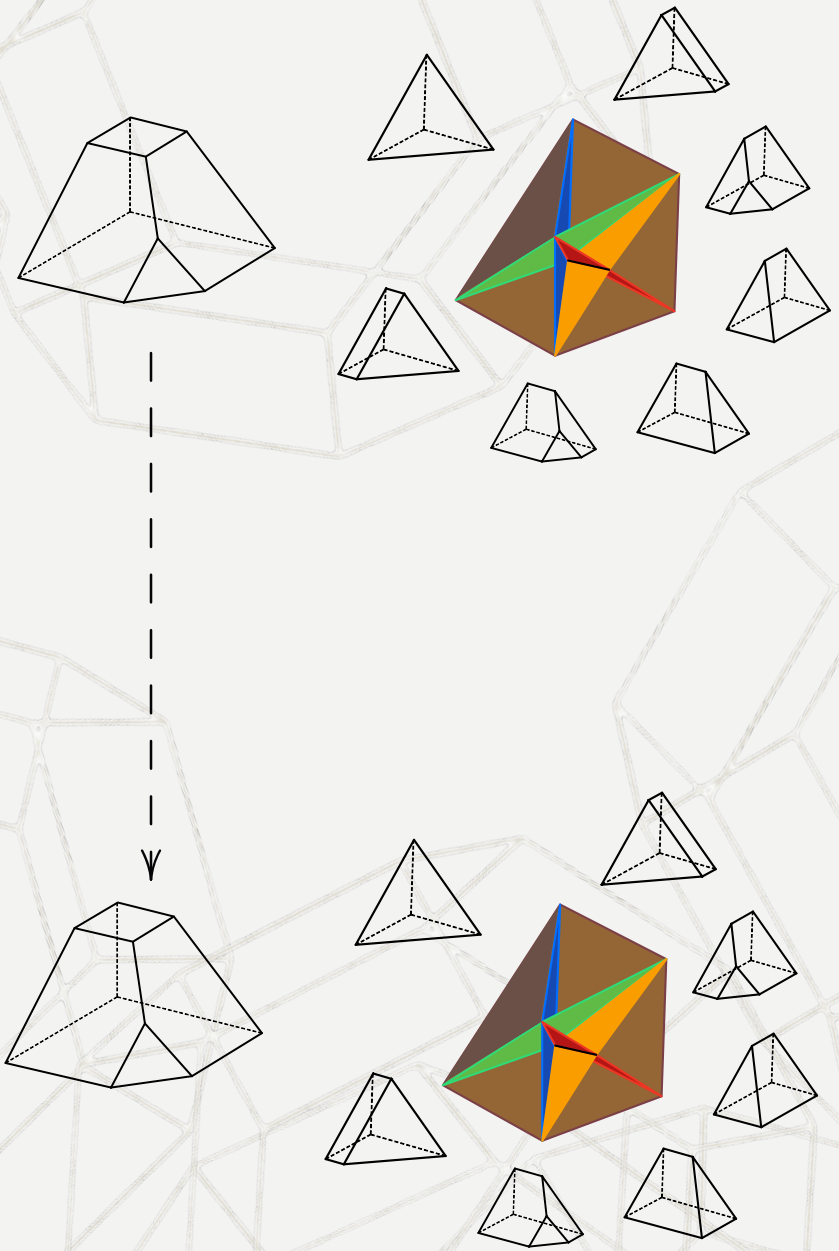


From $\mathcal{U}_e$ to $\mathcal{X}$





The tile group

- This is $W(\tau)$ as a group of bir. aut. of $\mathbb{P}^3$
- Generators R_1, R_2, R_3, T of $W(\tau)$ in some adequate coordinates:

$$R_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$R_2([x_0:x_1:x_2:x_3]) = [(x_1-x_0)(x_2-x_0):x_1(x_2-x_0):x_2(x_1-x_0):x_1x_2-x_0x_3]$$

Image of the boundary divisors in $\mathbb{P}^3$

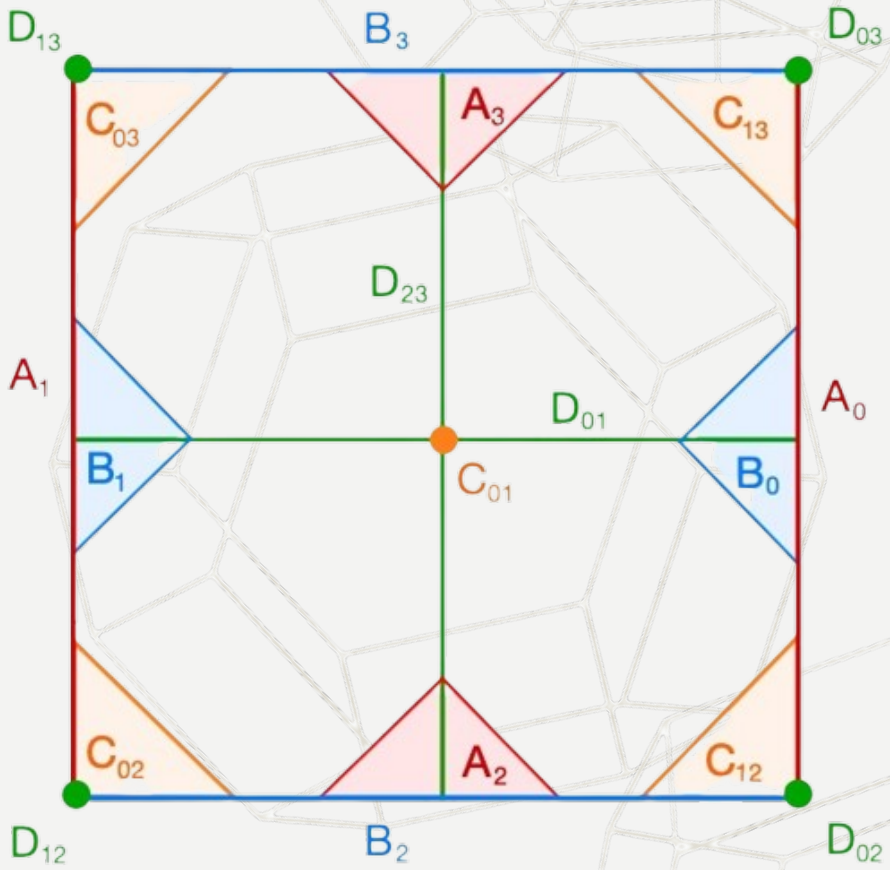
We can compute the transforms of the boundary divisors via the elements of the tile group:

A_0	line	$x_1 = x_3 = 0$	B_0	plane	$x_1 - x_3 = 0$
A_1	line	$x_0 = x_2 = 0$	B_1	plane	$x_0 - x_2 = 0$
A_2	plane	$x_0 - x_1 = 0$	B_2	line	$x_0 = x_1 = 0$
A_3	plane	$x_2 - x_3 = 0$	B_3	line	$x_2 = x_3 = 0$
C_{01}	point	$[1:1:1:1]$	D_{01}	line	$x_0 - x_2 = x_1 - x_3 = 0$
C_{02}	plane	$x_0 = 0$	D_{02}	point	$[0:0:1:0]$
C_{03}	plane	$x_2 = 0$	D_{03}	point	$[1:0:0:0]$
C_{12}	plane	$x_1 = 0$	D_{12}	point	$[0:0:0:1]$
C_{13}	plane	$x_3 = 0$	D_{13}	point	$[0:1:0:0]$
C_{23}	quadric	$x_0x_3 - x_1x_2 = 0$	D_{23}	line	$x_0 - x_1 = x_2 - x_3 = 0$

All points and lines are contained in the quadric C_{23}; the planes are tangent to it.

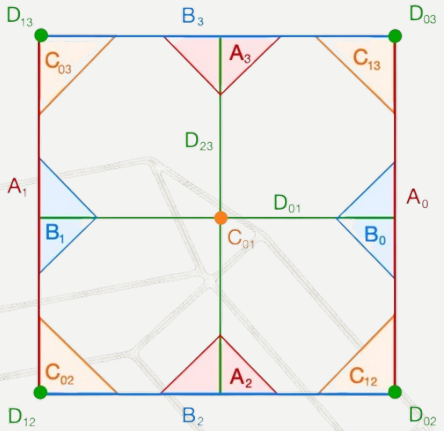
The tile

Numbering is compatible with W (=permutations of $\{0, 1, 2, 3\}$)



The tile 3-fold

$\widetilde{\mathcal{C}X}$ is defined as the successive blowup of $\mathbb{P}^3$ along $A_0 \sqcup A_1$, the strict transform of $B_2 \sqcup B_3$, C_{01}, and the strict transform of all the D's



- It admits contractions $\widetilde{\pi}_\sigma : \widetilde{\mathcal{C}X} \to \mathcal{C}U_\sigma$ commuting with the birational maps $\mathcal{C}U_w \dashrightarrow \mathcal{C}U'_w$, hence we get a surjective morphism $\psi : \widetilde{\mathcal{C}X} \to \mathcal{C}X$
- We check that the intersection of the exceptional loci of the morphisms $\{\widetilde{\pi}_\sigma \mid \sigma \in W\}$, is empty, so ψ cannot contract a curve
- Conclusion:

$$\boxed{\psi : \widetilde{\mathcal{C}X} \xrightarrow{\simeq} \mathcal{C}X}$$

- In particular $\mathcal{C}X$ is smooth

Birational Geometry of $\mathcal{C}X$

The Picard group of $\mathcal{C}X$

- A_i, B_i, C_{ij}, D_{ij} in $\mathcal{C}X$ are called boundary divisors; they generate

$$\mathrm{Pic}(\mathcal{C}X) \simeq \mathbb{Z}^{12}$$

- We may compute their relations by considering the total transform of the eight planes in the tile:

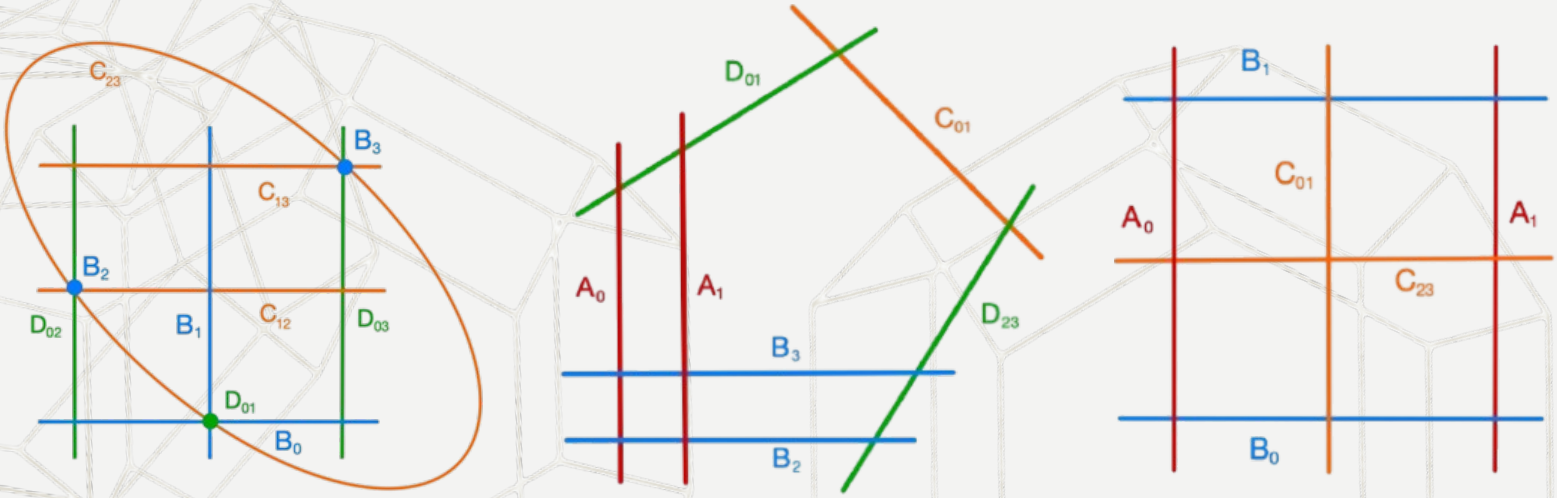
Plane	q^*H
A_2	$A_2 + B_2 + C_{01} + D_{02} + D_{12} + D_{23}$
A_3	$A_3 + B_3 + C_{01} + D_{03} + D_{13} + D_{23}$
B_0	$A_0 + B_0 + C_{01} + D_{01} + D_{02} + D_{03}$
B_1	$A_1 + B_1 + C_{01} + D_{01} + D_{12} + D_{13}$
C_{02}	$A_1 + B_2 + C_{02} + D_{02} + D_{12} + D_{13}$
C_{12}	$A_0 + B_2 + C_{12} + D_{02} + D_{12} + D_{03}$
C_{13}	$A_0 + B_3 + C_{13} + D_{02} + D_{03} + D_{13}$
C_{03}	$A_1 + B_3 + C_{03} + D_{03} + D_{12} + D_{13}$

and of the quadric C_{23}:

$$2q^*H = A_0 + A_1 + B_2 + B_3 + C_{01} + C_{23} + D_{01} + D_{02} + D_{03} + D_{12} + D_{13} + D_{23}$$

Geometry of the boundary divisors

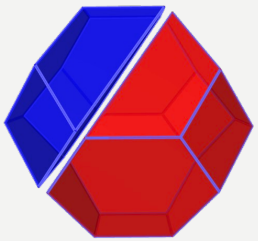
- A_i's, B_i's are del Pezzo surfaces of degree 5 (τ exchanges them)
- C_{ij}'s are del Pezzo surfaces of degree 7
- D_{ij}'s are $\mathbb{P}^1 \times \mathbb{P}^1$'s
- We represent here A_0, C_{23}, D_{01}:



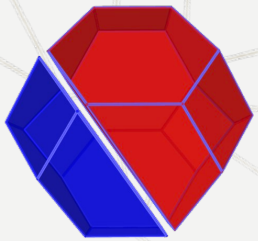
- We may use this to compute double and triple intersections of boundary divisors

Intersection theory

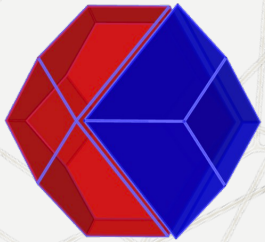
Divisors



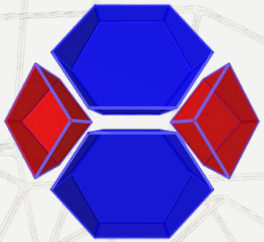
A



B

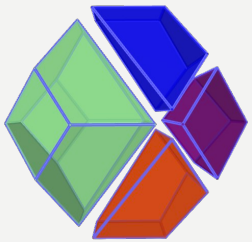


C

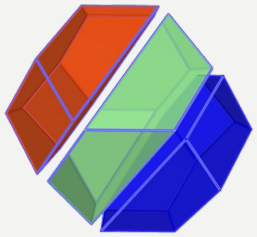


D

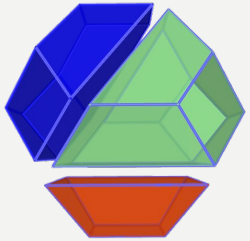
Curves



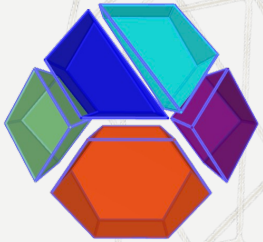
$A_i \star B_j$



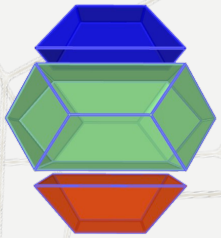
$A_i \star B_i$



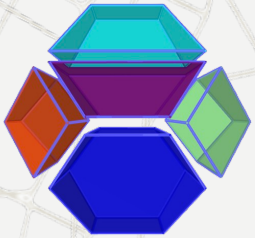
$A \star C$



$A \star D$

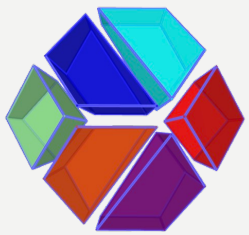


$C \star C$

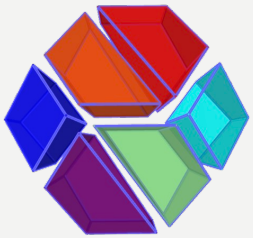


$C \star D$

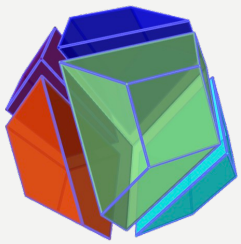
Points



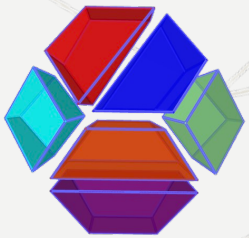
$A_i \star B_j \star D$



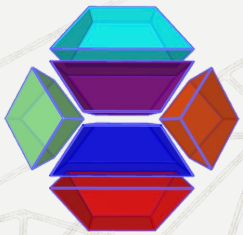
$A_i \star B_i \star D$



$A \star B \star C$



$A \star C \star D$



$C \star C \star D$

$\mathcal{C}X$ is an MDS

- Using the contraction $q : \mathcal{C}X \to \mathbb{P}^3$:

$$-K_X = 4q^*H - (A_0 + A_1 + B_2 + B_3 + D_{01} + D_{23} + 2(D_{02} + D_{03} + D_{12} + D_{13} + C_{01}))$$

- $|-K_X|$ is the linear system of quartics in $\mathbb{P}^3$ passing through the lines A_0, A_1, B_0, B_1, D_{01}, $D_{23} \subset \mathbb{P}^3$
- We may compute it explicitly and show that it is bpf, $\dim |-K_X| = 14$. In particular, $-K_X$ is nef
- Use intersection theory to compute

$$(-K_X)^3 = 12$$

Theorem

The Chow quotient X *is a weak Fano manifold, hence an MDS.*

The Mori cone of $\mathcal{C}X$ and its dual

Theorem

The cone $\overline{\mathrm{NE}(X)}$ *has* 31 *extremal rays, generated by intersections of two boundary divisors.*

- There are 12 K_X-negative rays: give contractions of the D_{ij}'s
- There are 19 K_X-trivial extremal rays: 12 of type $A_i \star B_j$; 4 of type $A_i \star B_i$; three of type $C_{ij} \star C_{k\ell}$
- The Nef cone of X has 189 extremal rays

Contractions

- 189 maximal contractions: 20 fiber type and 169-birational
- 9 contractions to $\mathbb{P}^1$ in two orbits $9 = 8 + 1$
- The unique $W(\tau)$-invariant contraction to $\mathbb{P}^1$ is the Chow quotient of $G(2,4) = F(2)$
- $11 = 1 + 2 + 8$ maximal contractions onto surfaces, all isomorphic to $\mathbb{P}^2$
- The only $W(\tau)$-invariant contraction to $\mathbb{P}^2$ is the Chow quotient of $F(1,3)$
- Studying $W(\tau)$-invariant contractions onto surfaces, we get that the Chow quotients of $F(1,2)$, $F(2,3)$: they are del Pezzo surfaces of degree 5

Open questions and future developments

- G semisimple, $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$: study the combinatorial quotient of $\mathfrak{n}_\pm$ by H ($\mathcal{C}\mathsf{U}$)
- Build tools to compute (birational) properties of $\mathcal{C}\mathsf{X}$ out of the $\mathcal{C}\mathsf{U}$'s without an explicit description of $\mathcal{C}\mathsf{X}$

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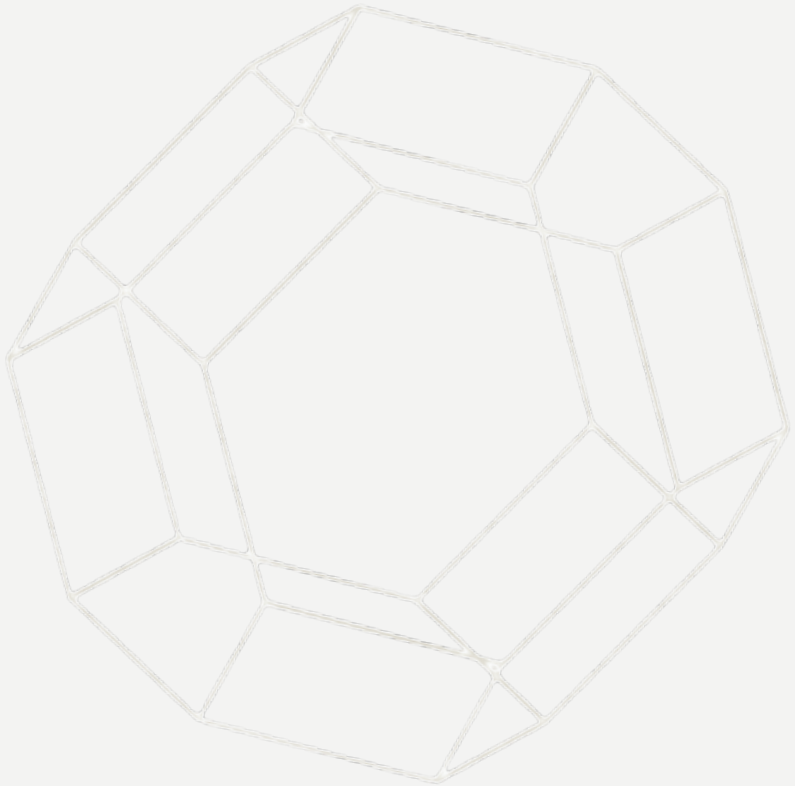

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